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(Basics of the) Fermi Gas Theory

M. Baranov

Institute for Quantum Optics and Quantum Information,
Institute for Theoretical Physics, University of Innsbruck

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Content of the lecture

Introduction:

- fermionic operators, their algebra and Pauli principle

Ideal Fermi gas:

- ground state, excitations, basic properties

Interacting Fermi gas:

- perturbative approach and Pauli blocking
- repulsive interaction: quasiparticles and Landau zero sound
- attractive interaction: BCS pairing

Conclusion:

- repulsive vs. attractive Fermi gas

Introduction: fermionic operators, their algebra and Pauli principle

Fermionic operators

a_ν, a_ν^+ - fermionic annihilation and creation operators of a **particle** in a quantum state ν

ν - complete set of quantum numbers which characterize a single-particle state

$\nu = (\vec{p}, \sigma)$ for a gas, $\vec{p}$ - momentum, σ - component/species index

Operator algebra

$$\{a_\nu, a_\mu^+\} = a_\nu a_\mu^+ + a_\mu^+ a_\nu = \delta_{\nu,\mu} \qquad \{a_\nu, a_\mu\} = 0 \qquad \{a_\nu^+, a_\mu^+\} = 0$$

$\Downarrow$ $\Downarrow$

Pauli principle $\quad (a_\nu)^2 = 0 \quad (a_\nu^+)^2 = 0$

Ideal Fermi gas: ground state, excitations, basic properties

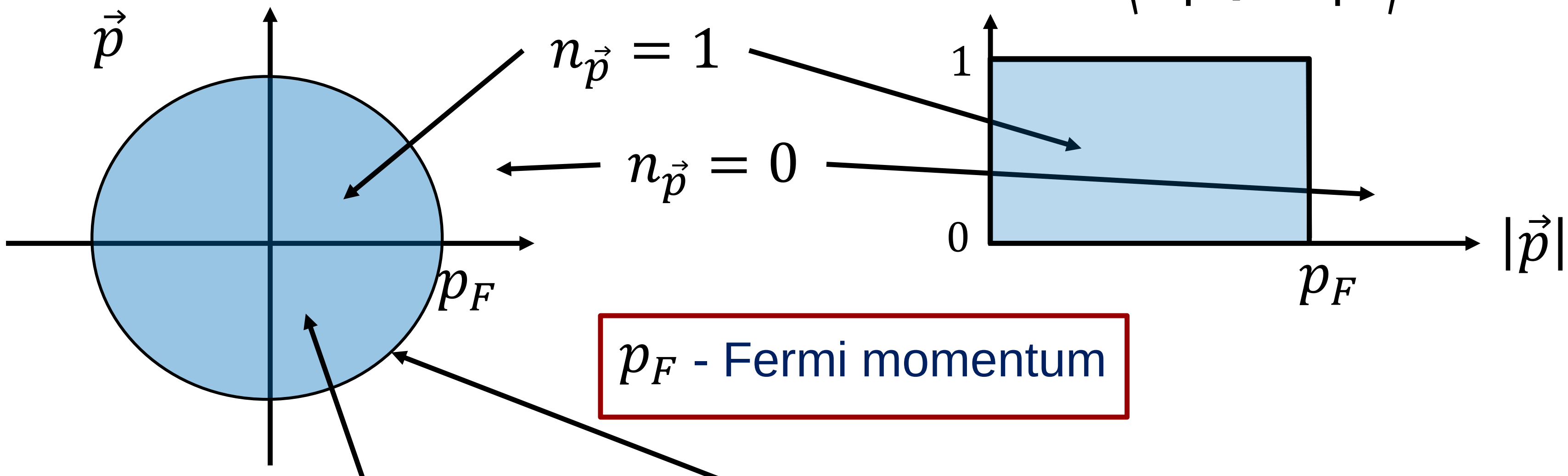
N identical fermions (single-component gas) in a volume V

$N \to \infty, V \to \infty, N/V = n$ - (concentration) fixed

Hamiltonian $\hat{H}_0 = \Sigma_{\vec{p}} \varepsilon_p a^+_{\vec{p}} a_{\vec{p}}$, $\varepsilon_p = \dfrac{p^2}{2m}$

Ground state: $|G\rangle = \Pi_{p \le p_F} a^+_{\vec{p}} |0\rangle$ - filled Fermi sphere (3D)

$$n_{\vec{p}} = \langle G | a^+_{\vec{p}} a_{\vec{p}} | G \rangle$$



p_F - Fermi momentum

Fermi sphere $p \le p_F$, Fermi surface $p = p_F$

What determines the Fermi momentum p_F ?

$$N = \Sigma_{\vec{p}} n_{\vec{p}} = \Sigma_{p \le p_F} 1 = V \int_{p \le p_F} \frac{d\vec{p}}{(2\pi\hbar)^3} = V \frac{4\pi}{(2\pi\hbar)^3} \frac{p_F^3}{3}$$

This gives

$$p_F = \hbar (6\pi^2 n)^{1/3}$$

depends only on the concentration n

Properties of the ground state

- energy: $E_0 = N \frac{3}{5} \varepsilon_F$ (kinetic energy!)

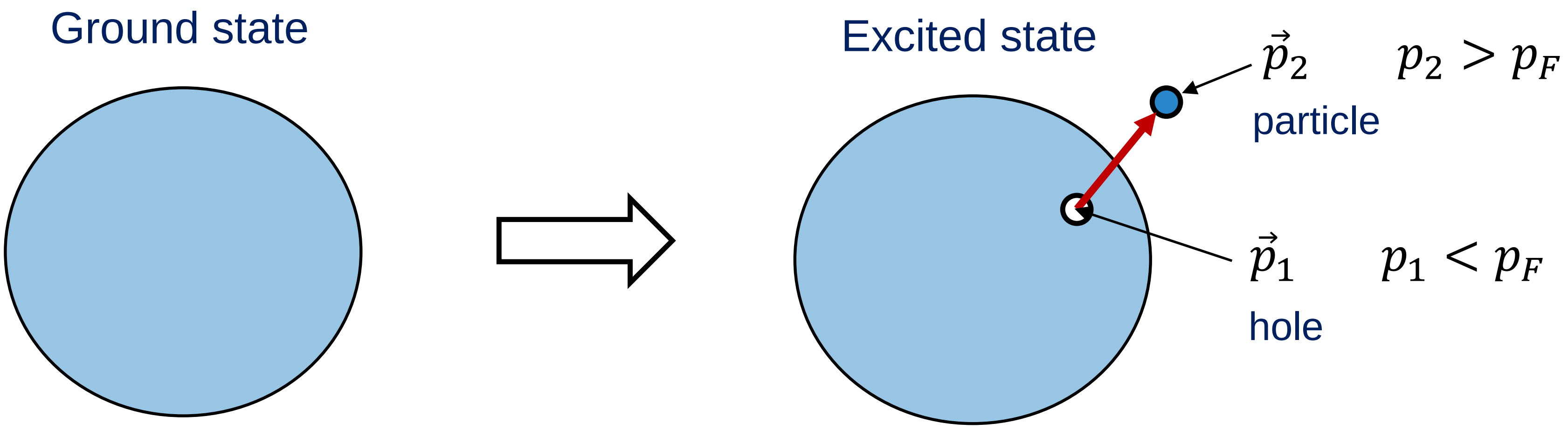
$$\varepsilon_F = \frac{p_F^2}{2m}$$ - Fermi energy

- pressure: $p = -\frac{\partial E_0}{\partial V} = \frac{2}{5} n \varepsilon_F \neq 0$ Fermi pressure

- chemical potential: $\mu = \frac{\partial E_0}{\partial N} = \varepsilon_F$

- density of state (DOS) at ε_F: $\nu_F = V^{-1} \Sigma_{\vec{p}} \delta\left(\varepsilon_p - \varepsilon_F\right) = \frac{m p_F}{2\pi^2 \hbar^3}$

Excitations:



particle excitation: $|\vec{p}\rangle = a^+_{\vec{p}}|G\rangle$ ($\neq 0$ only for $p > p_F$ and has $N+1$ particles!)

$$E_p = \varepsilon_p + \underbrace{E_0(N) - E_0(N+1)}_{= -\mu} = \varepsilon_p - \mu = \varepsilon_p - \varepsilon_F > 0$$

hole excitation: $|\vec{p}\rangle = a_{-\vec{p}}|G\rangle$ ($\neq 0$ only for $p < p_F$ and has $N-1$ particles!)

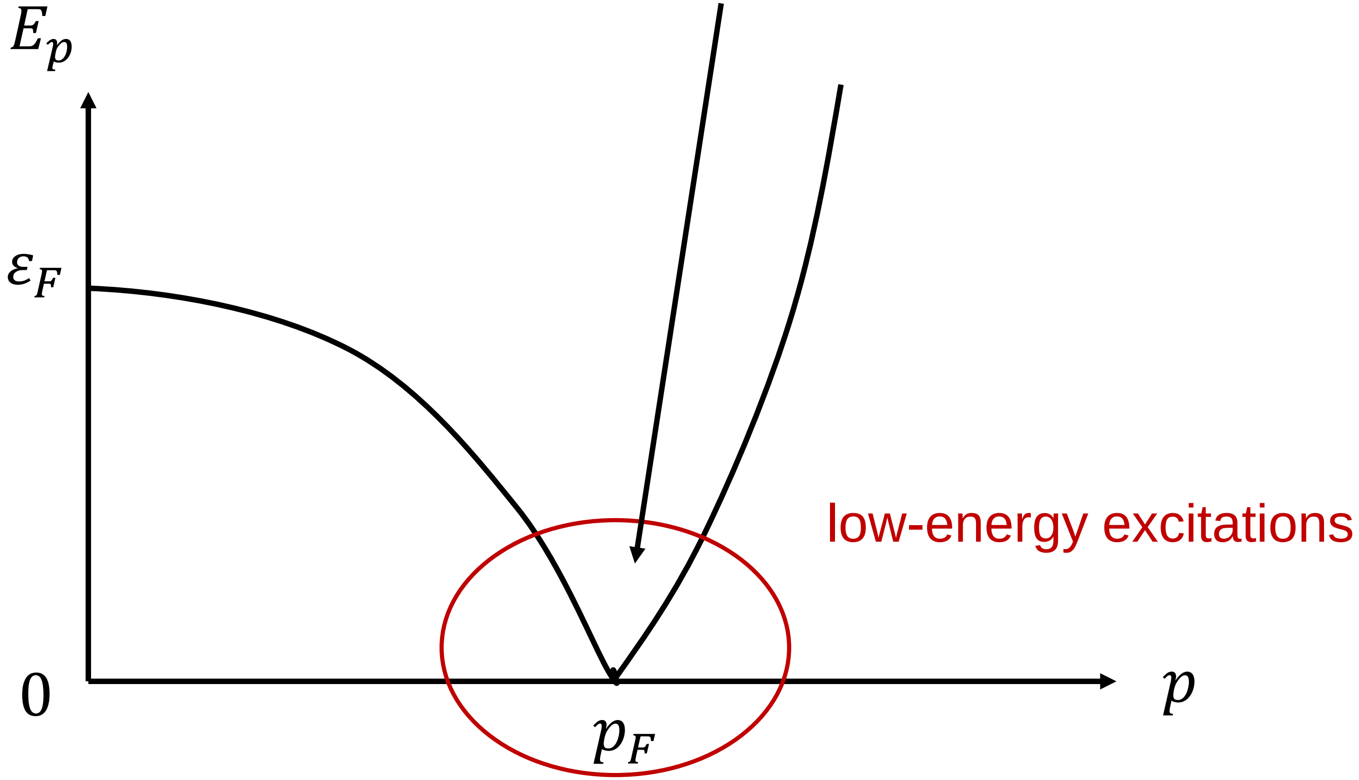
$$E_p = -\varepsilon_p + \underbrace{E_0(N) - E_0(N-1)}_{= \mu} = \mu - \varepsilon_p = \varepsilon_F - \varepsilon_p > 0$$

General expression:

$$E_p = |\varepsilon_p - \varepsilon_F| = \left|\frac{p^2 - p_F^2}{2m}\right| \approx \frac{p_F}{m}|p - p_F|$$

gapless!

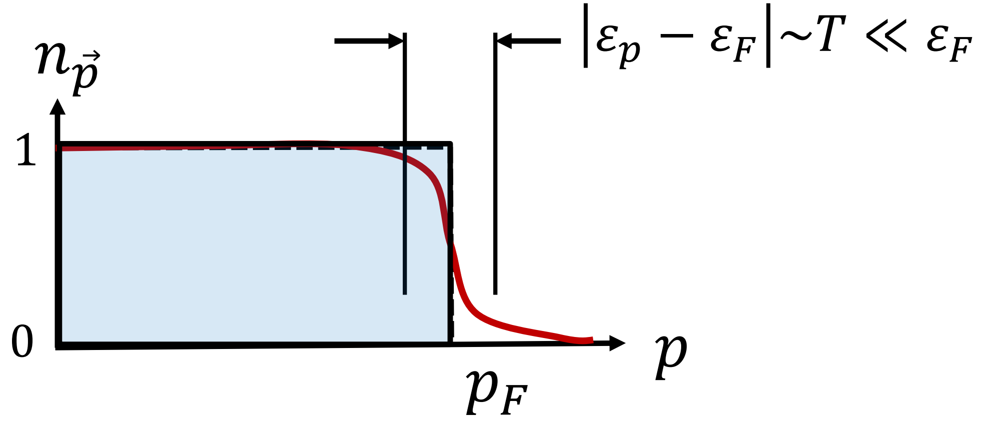
for $p \approx p_F$



Finite (small) temperature $T \ll \varepsilon_F = T_F$ $(k_B = 1)$

Only excitations with $p \approx p_F$ are relevant !

$$n_p = \frac{1}{e^{(\varepsilon_p - \mu)/T} + 1}$$



At such temperatures:

- energy: $E(T) = E_0 + (\pi^2/6)\nu_F T^2$
- specific heat: $c_V = (\pi^2/3)\,\nu_F T$ Signature of the Fermi sphere !
- chemical potential: $\mu = \varepsilon_F \left(1 - \frac{\pi^2}{12}\frac{T^2}{{\varepsilon_F}^2}\right)$

Interacting Fermi gas: Perturbative approach and Pauli blocking

$V(\vec{r})$ interparticle interaction with the range r_0

Gas: $n{r_0}^3 \ll 1$

Ultracold gas: (quantum) $\frac{\hbar}{p} \gg n^{-1/3}$

p - typical momentum

$\Rightarrow$ $pr_0/\hbar \ll 1$

s-wave scattering only !

$\Downarrow$

1. We need at least two components (species) to see the effects of the interaction

2. Interaction can be simplified as

$$V(\vec{r}) \to \text{g}\delta(\vec{r}) \quad \text{with} \quad \text{g} = \frac{4\pi\hbar^2}{m} a_s$$

a_s - s-wave scattering length

Two-component Fermi gas: $\sigma = \pm, m_\pm = m, n_\pm = n$

Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{H}_{int} = \hat{H}_0 + \mathrm{g}\int d\vec{r}\, n_+(\vec{r}) n_-(\vec{r})$$

$$= \Sigma_{\vec{p},\sigma} \varepsilon_p a^+_{\vec{p},\sigma} a_{\vec{p},\sigma} + \frac{\mathrm{g}}{V}\Sigma_{\vec{p}_1,\vec{p}_2,\vec{q}}\, a^+_{\vec{p}_1+\vec{q},+} a^+_{\vec{p}_2-\vec{q},-} a_{\vec{p}_2,-} a_{\vec{p}_1,+}$$

First order interaction effects:

$$E = E_0 + \langle G|\hat{H}_{int}|G\rangle = V\left\{2n\frac{3}{5}\epsilon_F + \mathrm{g}n^2\right\}$$

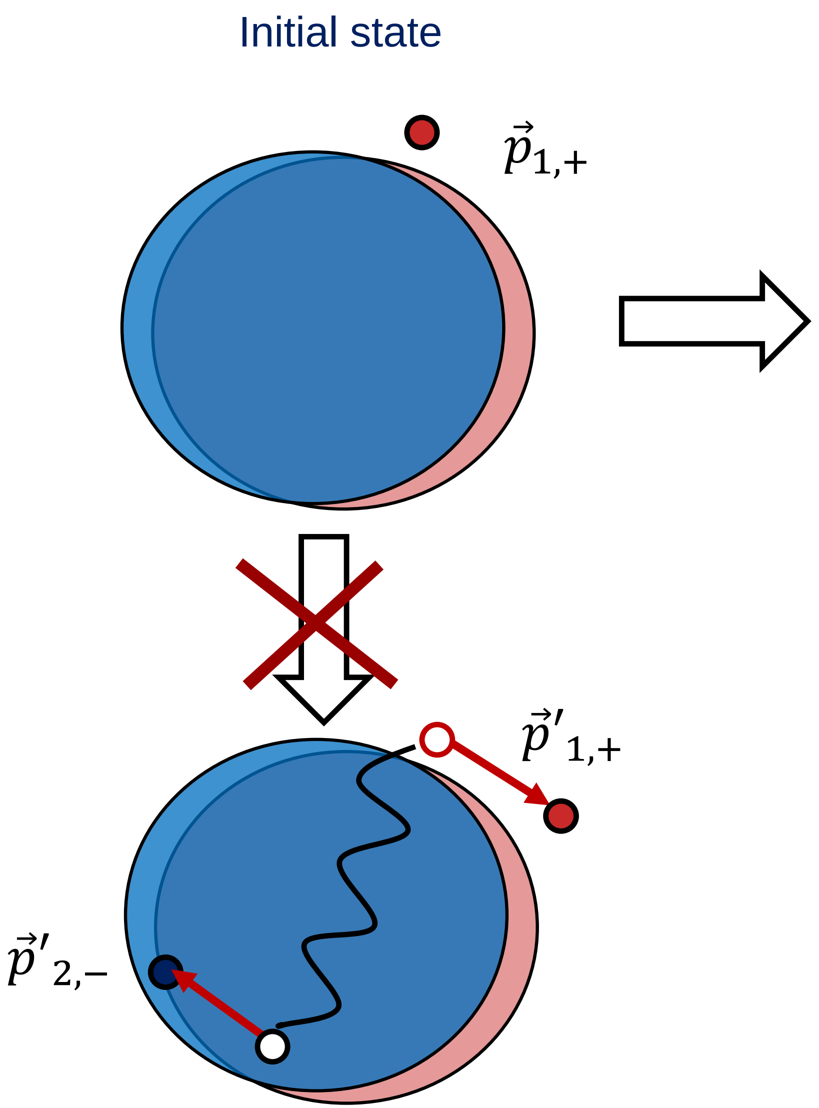
$$\mu_\pm = \epsilon_F + \mathrm{g}n_\mp = \epsilon_F + \mathrm{g}n = \mu$$

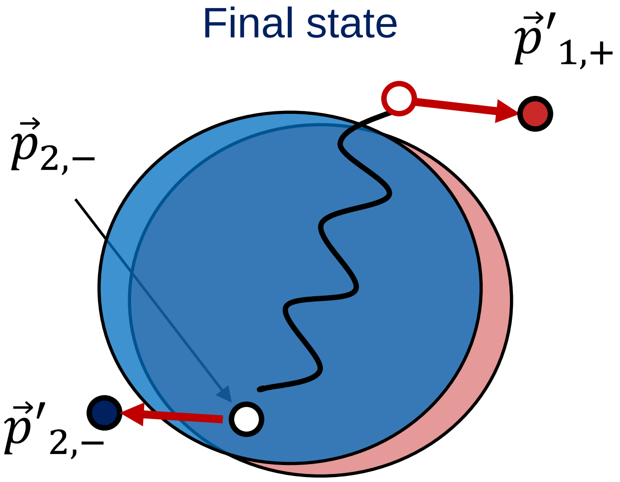
Small parameter for perturbative calculations:

$$\frac{\langle G|\hat{H}_{int}|G\rangle}{E_0} \sim \frac{a_s p_F}{\hbar} \sim a_s n^{1/3} \ll 1$$

Pauli blocking for scattering:

Pauli principle strongly reduces available final states for scattering of particles **near the Fermi surface**





Pauli principle

$p_{1,+} > p_F$

$p'_{1,+} > p_F$

$p_{2,-} < p_F$

$p'_{2,-} > p_F$

$$\vec{p}_{1,+} + \vec{p}_{2,-} = \vec{p}'_{1,+} + \vec{p}'_{2,-}$$

$$\varepsilon_{1,+} + \epsilon_{2,-} = \varepsilon_{1',+} + \varepsilon_{2',-}$$

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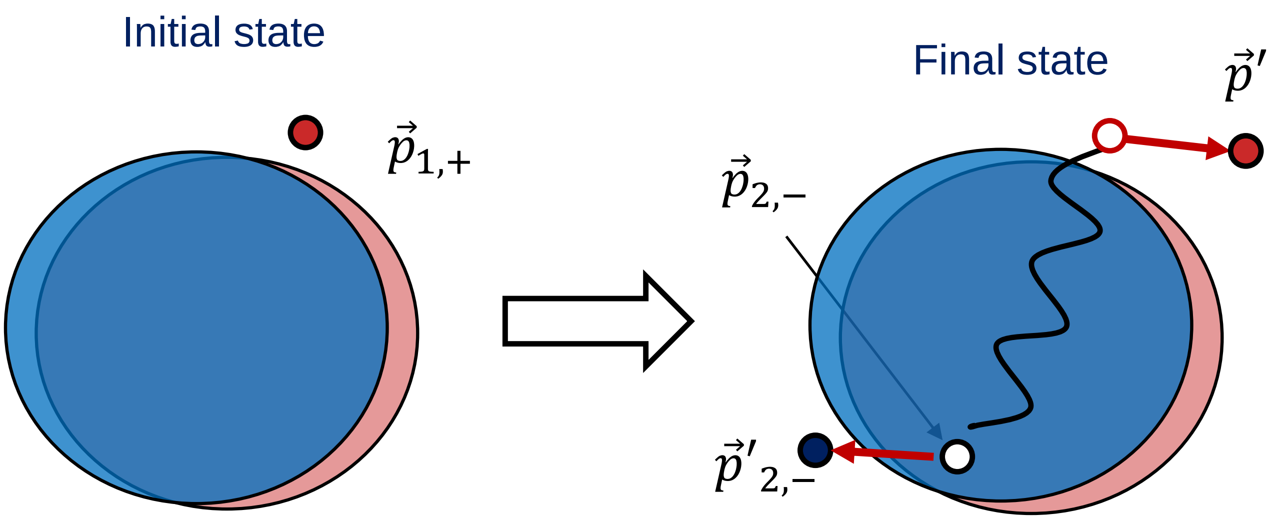

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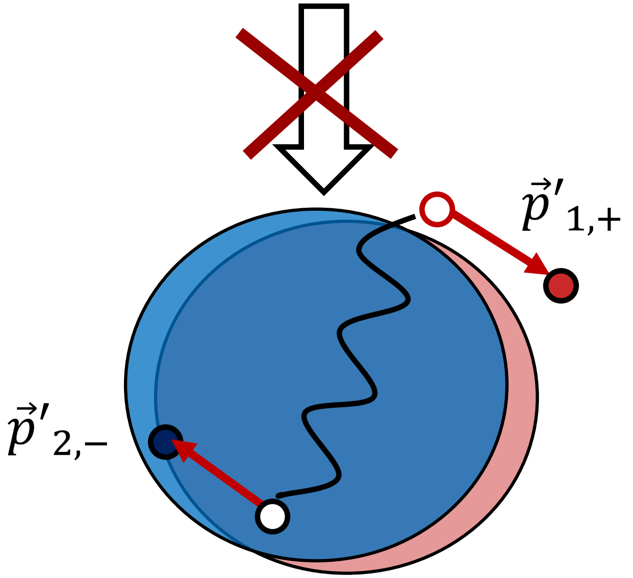


$$\vec{p}_{1,+} + \vec{p}_{2,-} = \vec{p}'_{1,+} + \vec{p}'_{2,-}$$

$$\varepsilon_{1,+} + \epsilon_{2,-} = \varepsilon_{1',+} + \varepsilon_{2',-}$$

or

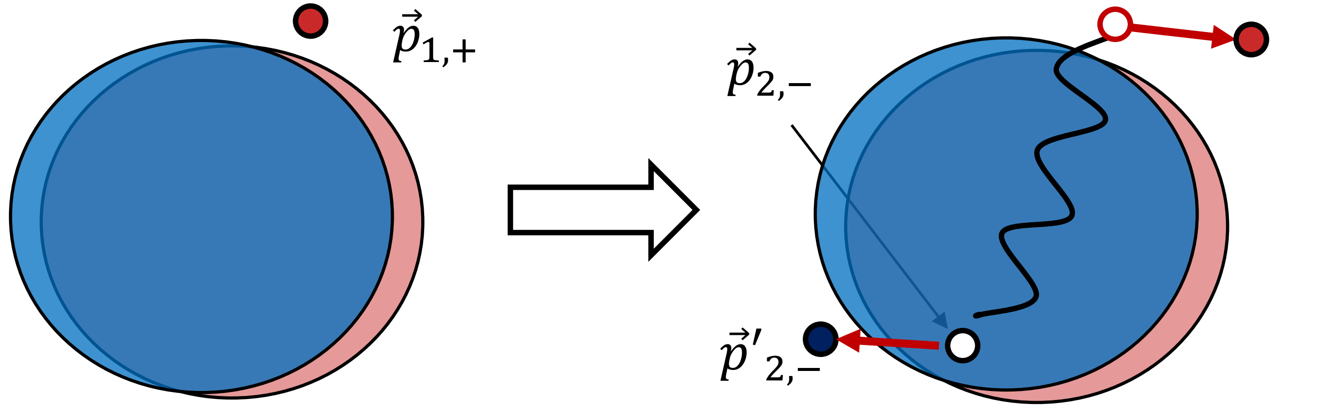
$$(\varepsilon_{1,+} - \varepsilon_F) = (\varepsilon_{1',+} - \varepsilon_F) + (\varepsilon_{2',-} - \varepsilon_F) + (\varepsilon_F - \epsilon_{2,-})$$

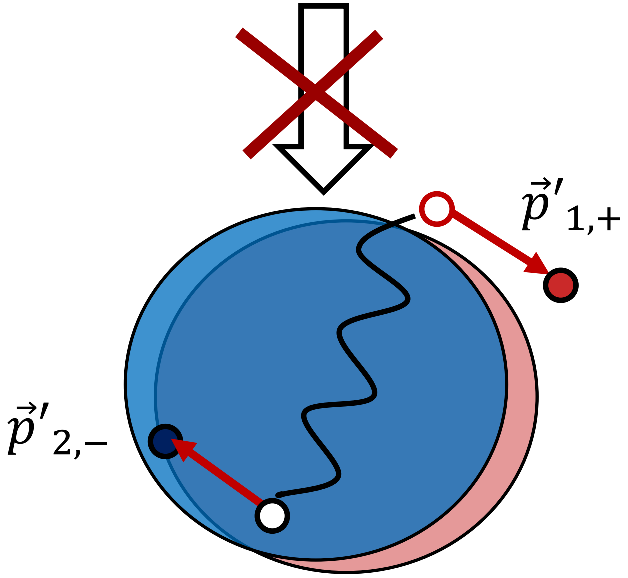
Pauli blocking for scattering:

Pauli principle strongly reduces available final states for scattering of particles **near the Fermi surface**

Initial state

Final state





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or

$$(\varepsilon_{1,+} - \varepsilon_F) = (\varepsilon_{1',+} - \varepsilon_F) + (\varepsilon_{2',-} - \varepsilon_F) + (\varepsilon_F - \epsilon_{2,-})$$

$> 0_F$ | $> 0_F$ | $> 0_F$ | $> 0_F$

Pauli blocking !

As a result of Pauli blocking for final states,

the life-time τ_p of the excitation with momentum p

$$\frac{1}{\tau_p} = \frac{1}{\tau_{cl}} \boxed{max\left\{\left(\frac{E_p}{\varepsilon_F}\right)^2, \left(\frac{T}{\varepsilon_F}\right)^2\right\}} \ll \frac{1}{\tau_{cl}}$$

Pauli blocking !

where $\tau_{cl} \sim n a_s^2 \frac{p_F}{m}$ - classical collisional time

Consequences:

1. Excitations close to the Fermi surface, $p \approx p_F$, are long-lived and, therefore, are well-defined (quasiparticles)

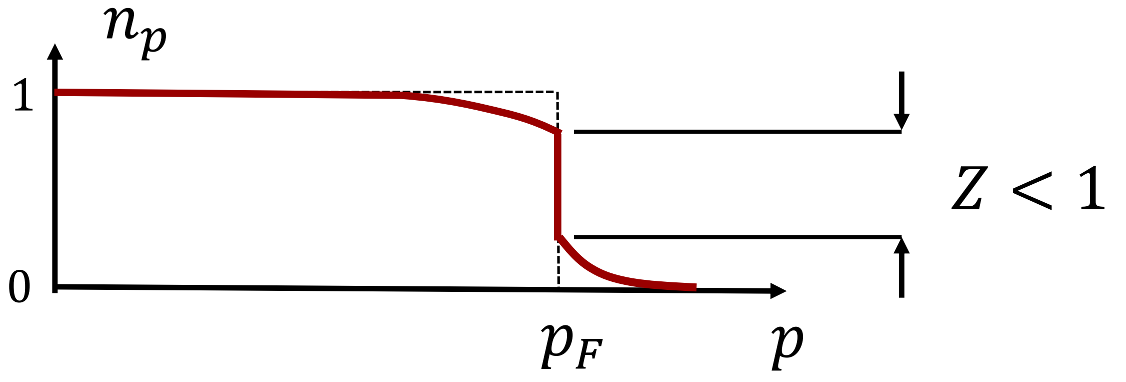
2. One needs a long time to reach local equilibrium. As a result, hydrodynamic regime $\omega\tau \ll 1$ is questionable

Two-component Fermi gas with repulsive interaction $g > 0$

$(a_s > 0)$

Results of the interaction: **Fermi-liquid renormalization of single-particle excitations** (particles and holes) and appearance of the collective mode – **Landau zero sound**:

1. Particle momentum distribution



n_p is discontinuous at $p = p_F$

$$Z = 1 - \frac{8 \ln 2}{\pi^2} \left(\frac{a_s p_F}{\hbar} \right)^2$$

2. Single-particle excitations for $p \approx p_F$

$$E_p \approx \frac{p_F}{m_*}|p - p_F|$$

with **effective mass**

$$m_* = m\left[1 + \frac{8}{15\pi^2}(7\ln 2 - 1)\left(\frac{a_s p_F}{\hbar}\right)^2\right]$$

Single-particle properties are similar to those of an **ideal gas of quasiparticles** with the effective mass m_* and the same density n (same p_F).

3. Collective mode – Landau zero sound

(coherent motion of particle-hole excitations)

$$\omega_k = ck \quad \text{with} \quad c \approx \frac{p_F}{m_*}\left[1 + 2\,exp\left(-\frac{\pi\hbar}{a_s p_F}\right)\right]$$

It is a “high-frequency” sound $\omega_k \tau \gg 1$
without establishing local equilibrium (**not hydrodynamic sound !**)

Two-component Fermi gas with attractive interaction $g < 0$

$(a_s < 0)$

Sharp Fermi surface is unstable against Cooper pairing

In the new ground state $|\tilde{G}\rangle$

$$\Delta = \frac{g}{V} \Sigma_{\vec{p}} \langle \tilde{G} | a_{-\vec{p},-} a_{\vec{p},+} | \tilde{G} \rangle \neq 0$$

- order parameter

BCS Hamiltonian

comes from the interaction term

$$\hat{H}_{BCS} = \Sigma_{\vec{p},\sigma} (\varepsilon_p - \mu) a^+_{\vec{p},\sigma} a_{\vec{p},\sigma} + \Delta \Sigma_{\vec{p}} (a^+_{\vec{p},+} a^+_{-\vec{p},-} + a_{-\vec{p},-} a_{\vec{p},+})$$

Pairs of particles with opposite momentum **appear/disappear** from/to the collective degree of freedom Δ

Diagonalization via Bogoliubov transformation

We define quasiparticle (excitation) fermionic operators $\alpha_{\vec{p},\sigma}, \alpha^{+}_{\vec{p}\sigma}$

$$\left\{\alpha_{\vec{p},\sigma}, \alpha^{+}_{\vec{q},\rho}\right\} = \delta_{\vec{p},\vec{q}}\delta_{\sigma,\rho}$$

by the Bogoliubov transformation

$$\alpha_{\vec{p},+} = u_p a_{\vec{p},+} + v_p a^{+}_{-\vec{p},-} \qquad \alpha_{\vec{p},-} = u_p a_{\vec{p},-} - v_p a^{+}_{-\vec{p},+}$$

$$\alpha^{+}_{\vec{p},+} = u_p a^{+}_{\vec{p},+} + v_p a_{-\vec{p},-} \qquad \alpha^{+}_{\vec{p},-} = u_p a^{+}_{\vec{p},-} - v_p a_{-\vec{p},+}$$

with

$${u_p}^2 + {v_p}^2 = 1$$

Diagonalization via Bogoliubov transformation

With the choice

$$u_p^2 = \frac{1}{2}\left[1 + \frac{\varepsilon_p - \mu}{E_p}\right] \quad \text{and} \quad v_p^2 = \frac{1}{2}\left[1 - \frac{\varepsilon_p - \mu}{E_p}\right]$$

the Hamiltonian becomes diagonal:

$$\boxed{\hat{H}_{BCS} = \tilde{E}_0 + \Sigma_{\vec{p},\sigma} E_p \alpha^+_{\vec{p},\sigma} \alpha_{\vec{p},\sigma}}$$

with

excitation energy $$E_p = \sqrt{(\varepsilon_p - \mu)^2 + \Delta^2} > 0$$ will be discussed later

and

order parameter (gap) Δ

Equation for Δ (**gap equation**)

$$\Delta = \frac{4\pi\hbar^2}{m}\,|a_s| \int \frac{d\vec{p}}{(2\pi\hbar)^3}\left[\frac{\tanh(E_p/2T)}{2E_p} - \frac{m}{p^2}\right]\Delta$$

BCS model only!

Non-trivial solution $\Delta \neq 0$ exists only for $T < T_c$

Critical temperature T_c

$$T_c^{BCS} = \frac{e^\gamma}{\pi} 8e^{-2}\,\varepsilon_F e^{-1/\lambda} = 0.61\,\varepsilon_F e^{-1/\lambda}$$

$$T_c^{gas} = \frac{e^\gamma}{\pi}\left(\frac{2}{e}\right)^{7/3}\varepsilon_F e^{-1/\lambda} = 0.28\,\varepsilon_F e^{-1/\lambda}$$

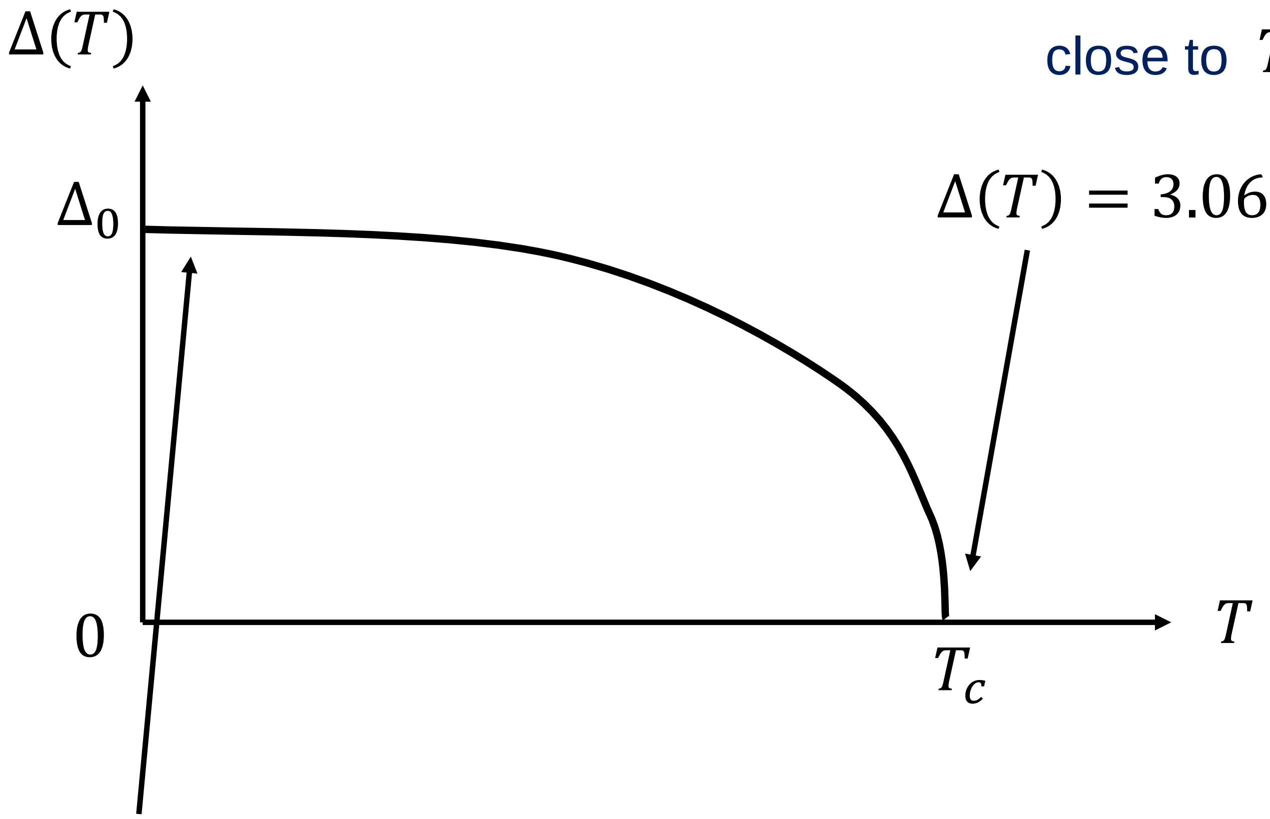
$T_c \neq 0$

for **any** $a_s < 0$

with

$$\lambda = \frac{2|a_s|p_F}{\pi\hbar} \ll 1 \qquad \gamma = 0.5772$$ - Euler constant

The order parameter $\Delta(T)$



close to T_c

$$\Delta(T) = 3.06\, T_c \sqrt{1 - T/T_c}$$

$$\Delta_0 = \pi e^{-\gamma} T_c = 1.57\, T_c$$ at small temperatures $T \to 0$

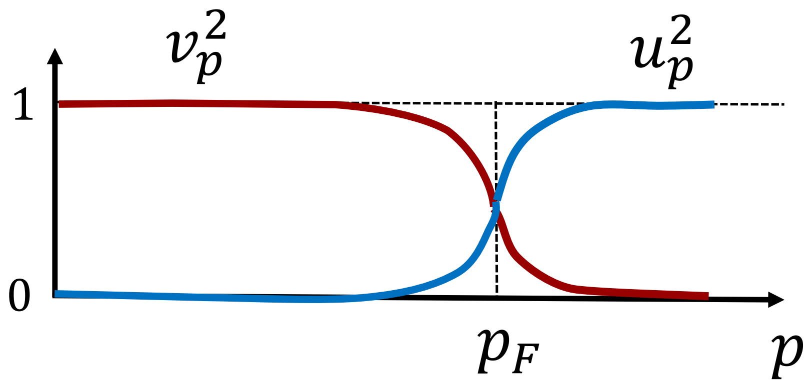
Second order phase transition!

New ground state $|\tilde{G}\rangle$

$$\alpha_{\vec{p},\sigma}|\tilde{G}\rangle = 0 \quad \text{for all} \quad \alpha_{\vec{p},\sigma}$$

Solution

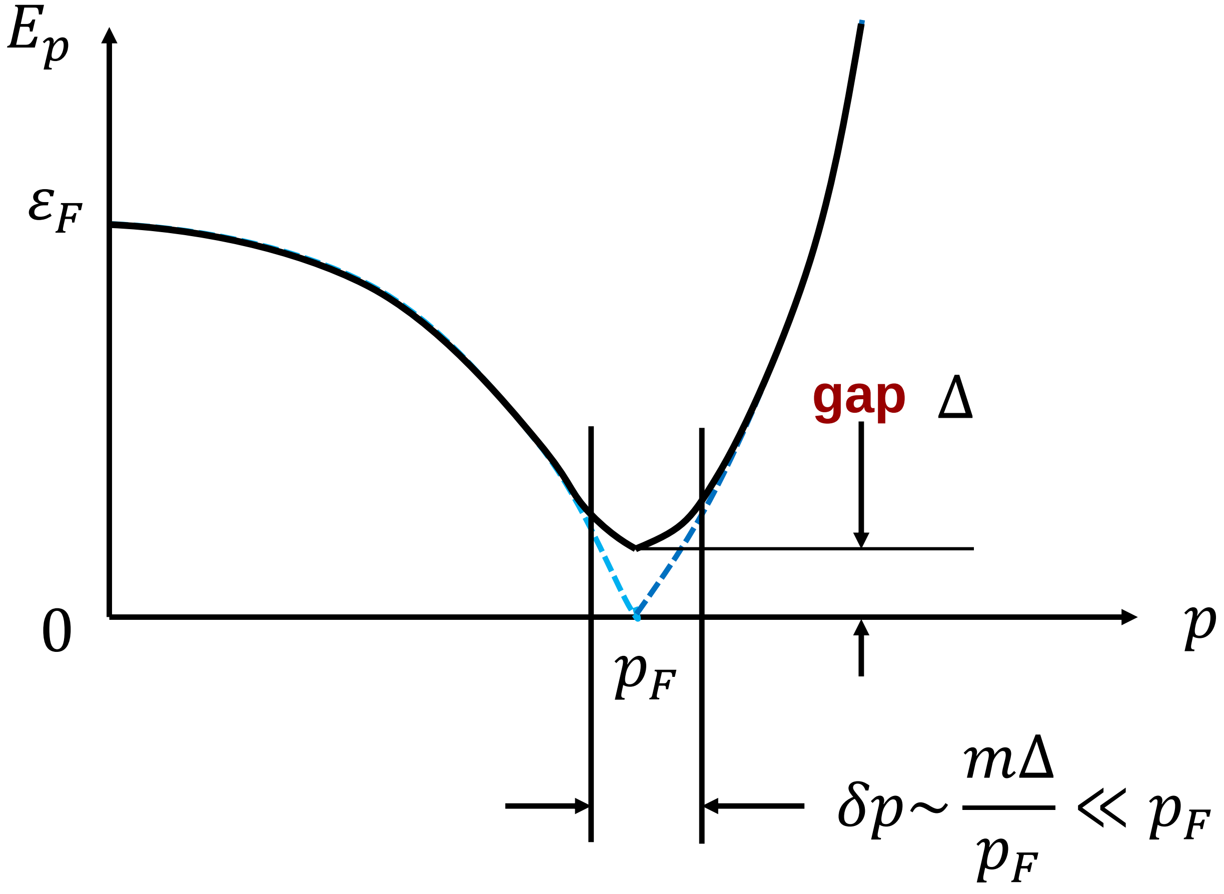
$$|\tilde{G}\rangle = \Pi_{\vec{p}}(u_p + v_p a^+_{\vec{p},-} a^+_{-\vec{p},+})|0\rangle$$ - population in pairs



Particle momentum distribution

$$n_{\vec{p},\sigma} = \left\langle \tilde{G} \middle| a^+_{\vec{p},\sigma} a_{\vec{p},\sigma} \middle| \tilde{G} \right\rangle = v_p^2$$ - continuous

Excitation energy $E_p = \sqrt{(\varepsilon_p - \mu)^2 + \Delta^2}$



Coherence length $\xi \sim \frac{\hbar}{\delta p} \sim \frac{\varepsilon_F}{\Delta} n^{-1/3} \gg n^{-1/3}$

strongly overlapping Cooper pairs

Collective excitations (low energy)

Phase fluctuations of the order parameter

$$\Delta \to \Delta(\vec{r}, t) = \Delta e^{i\varphi(\vec{r},t)}$$

correspond to the Bogoliubov-Anderson sound

$$\omega_k = ck \quad \text{with} \quad c \approx \frac{1}{\sqrt{3}} \frac{p_F}{m_*}$$

With gapped single-particle excitations and
sound-like collective excitations, the system is **superfluid** !

Conclusion:

normal ($a_s > 0$) vs. superfluid ($a_s < 0$) Femi gas

1. momentum distribution of particle (zero temperature)

   discontinuous — continuous

2. spectrum of single-particle excitations

   gapless — gapped (Δ)

3. collective excitations

   Landau zero sound $c \approx \frac{p_F}{m}$ — Bogoliubov-Anderson sound $c \approx \frac{p_F}{\sqrt{3}m}$

4. single-particle excitation density of states near Fermi surface

   finite — gapped (2Δ)

5. specific heat at low temperature (main contribution)

   $c_V \sim T$ (single-particle) — $c_V \sim T^3$ (collective)

Thank you for your attention!